

Leibniz-Dirac structures and dissipative Hamiltonian systems with constraints

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Abstract

Although conservative Hamiltonian systems with constraints can be formulated in terms of Dirac structures, a more general framework is necessary to cover also dissipative systems and gradient systems with constraints. We define Leibniz-Dirac (LD) structures which lead to a natural generalization of Dirac and Riemannian structures, for instance. From modeling point of view, LD structures make it easy to formulate implicit dissipative Hamiltonian systems. We give their exact characterization in terms of bundle maps from the cotangent bundle into the tangent bundle. Their behavior under push-forward maps is also considered. Physical systems which can be formulated in terms of LD structures are discussed.

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1 Introduction

Dirac structures embody a number of geometric structures such as symplectic, Poisson, foliation, complex geometries [1, 2]. Since their first introduction [3, 4] there have been a great number of work, which is still growing, have been done over the years. One of the most striking features of Dirac structures is that they can give a geometric picture of Hamiltonian systems with constraints, holonomic or nonholonomic

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[5, 6]. Nevertheless, Dirac structures are insufficient in formulating non-conservative Hamiltonian systems such as gradient systems or dissipative systems. In that sense, recently some attempts have been done to put these systems into a rather Hamiltonian form.

For example, in [7], a generalization of Dirac structures is given in terms of an inner product of split sign on the Pontryagin bundle instead of the natural symmetric pairing. We specify this definition in order to cover the physical examples which are aimed to be put into the Hamiltonian context. In [8], the authors use the notion of Leibniz structures [9] which is a generalization of Poisson structures, whose tensor is not necessarily skew-symmetric. Our approach is quite similar but we both work on the Pontryagin bundle and with systems with constraints on the manifold. In [10], dissipative Hamiltonian systems with constraints are studied with Dirac's original method of reduced brackets. For other recent work on the generalizations of the conservative Hamiltonian systems we refer to [11, 12, 13] and the references therein. Our motivation in this paper is to give a generalization of Dirac structures and study their geometric features in order to construct a general framework of non-conservative Hamiltonian systems.

We define Leibniz-Dirac structures (LD structures in short) by weakening the properties of Dirac structures as follows: Let V be a vector space with its dual denoted by V^* . A subspace $\mathcal{D} \subset V \oplus V^*$ is called a Dirac structure if it is maximally isotropic under the symmetric pairing

$$\langle (v_1, v_1^*), (v_2, v_2^*) \rangle_+ = \langle v_1^* | v_2 \rangle + \langle v_2^* | v_1 \rangle$$

for all $(v_1, v_1^*), (v_2, v_2^*) \in V \oplus V^*$, where $\langle | \rangle$ denotes the natural pairing between vectors and co-vectors. As a result, it is shown in [1] for a Dirac structure that the equations, which are called *characteristic equations*,

$$\rho(\mathcal{D})^\circ = \mathcal{D} \cap V^* \text{ and } \rho^*(\mathcal{D}) = (\mathcal{D} \cap V)^\circ$$

are satisfied, where ρ and ρ^* denote the projections from $V \oplus V^*$ onto the first and second factor respectively, and $(^\circ)$ stands for the annihilation operator. Accordingly, there exist skew-symmetric linear maps $\omega : E = \rho(\mathcal{D}) \rightarrow E^*$ and $J : \rho^*(\mathcal{D}) \rightarrow \rho^*(\mathcal{D})^*$. As the systems we aim to formalize can be given in terms of linear maps of the form $V^* \rightarrow V$, we assume the second characteristic equation as the defining conditions of the generalizations of Dirac structures. They are called Leibniz-Dirac structures (LD structures for short).

LD structures share some properties of Dirac structures such as being a Lagrangian subspace for a suitable inner product. Their extensions on manifolds is defined the same way as in the Dirac structures. Several properties of linear and smooth LD structures are discussed. It is shown that LD structures are general enough to express possibly dissipative implicit Hamiltonian systems with constraints. From the discussion above, it is seen that LD structures can be pushed forward but cannot be pushed back, in the sense of [14]. We define LD maps which transform LD structures to LD structures.

The paper is organized as follows. In Section 2 we define LD structures on manifolds and give their characterization in terms of vector bundle maps from the cotangent bundle into the tangent bundle with Theorem 1. Then it is shown that LD

structures are Lagrangian subspaces with respect to suitable symmetric pairings. Smooth LD structures on manifolds are defined in Section 14, where we also relate LD structures to Leibniz structures. In Section 3 we study admissible functions on manifolds with LD structures, then we study Hamiltonian dynamics of LD manifolds. We give present several physical examples which are also given in different formalisms. Maps that carry LD structures from one manifold to another manifold are studied in Section 4.2. The paper ends with some conclusions and future questions.

2 Leibniz-Dirac structures

In this section we develop the basic geometry of LD structures. First, we study linear LD structures and then extend it to smooth LD structures. Most of the results and notions are a refinement of the Dirac case [1].

2.1 Linear Leibniz-Dirac structures

Let V be an n -dimensional vector space and V^* be its dual space. Consider the direct product space $V \oplus V^*$ and denote the projections on V and V^* by ρ and ρ^* respectively. Also denote by W° the annihilator of a subspace $W \subset V$.

Before proceeding in definition of linear LD structures, we give the following elementary but key result on subspaces of $V \oplus V^*$ [1].

Proposition 1. *Let \mathcal{U} be subspace of $V \oplus V^*$, then the following equation holds:*

$$\dim(\mathcal{U} \cap V) + \dim(\rho^*(\mathcal{U})) = \dim(\mathcal{U}). \quad (1)$$

Proof. Let $\dim(\mathcal{U}) = k$. If one chooses a basis for \mathcal{U} , then this is equivalent to giving two linear maps $\mathbf{a} : \mathbb{R}^k \rightarrow V$ and $\mathbf{b} : \mathbb{R}^k \rightarrow V^*$ such that the basis becomes $(\mathbf{a} e_1, \mathbf{b} e_1), \dots, (\mathbf{a} e_k, \mathbf{b} e_k)$. Since \mathcal{D} is k -dimensional,

$$\ker a \cap \ker b = \{0\} \quad (2)$$

is satisfied. Observe that $\mathcal{D} \cap V = a(\ker \mathbf{b})$ and $\rho^*(\mathcal{D}) = \text{range } \mathbf{b}$. But, $\dim(a(\text{range } \mathbf{b})) = \dim(\text{range } \mathbf{b})$ by (2). Then by the fact that $\dim(a(\ker b)) + \dim(\text{range } b) = k$ it is seen that Equation (1) is justified. \square

Definition 1. *Let V be a vector space. A vector subspace $\mathcal{D} \subset V \oplus V^*$ is called a Leibniz-Dirac structure (LD structure for short) on V if*

$$\rho^*(\mathcal{D}) = (\mathcal{D} \cap V)^\circ \quad (3)$$

is satisfied.

Apparently, this is a generalization of Dirac structures. This and the other special examples will be clear after the following theorem. But we first give the following conclusion which plays an important role in what follows.

Proposition 2. *Let \mathcal{D} be a subspace of $V \oplus V^*$ with $\dim(V) = n$, then the following are satisfied:*

- (i) *If \mathcal{D} is a LD structure, then $\dim(\mathcal{D}) = n$.*
- (ii) *If \mathcal{D} is n -dimensional and $\rho^*(\mathcal{D}) \subset (\mathcal{D} \cap V)^\circ$, then \mathcal{D} is a LD structure on V .*

Proof. (i) follows from Proposition 1, since $\dim((\mathcal{D} \cap V)^\circ) = n - \dim(\mathcal{D} \cap V)$. To show (ii), observe that $\rho^*(\mathcal{D}) \subset (\mathcal{D} \cap V)^\circ$ implies $\rho^*(\mathcal{D}) = (\mathcal{D} \cap V)^\circ$ again by Proposition 1. \square

Theorem 1. *If \mathcal{D} is a LD structure on a vector space V , then there exists a linear map $\mathcal{L} : \rho^*(\mathcal{D}) \rightarrow V/(\mathcal{D} \cap V)$ whose kernel is $\mathcal{D} \cap V^*$.*

Conversely, given a subspace $E \subset V$ and a linear map $\mathcal{L} : E^\circ \rightarrow V/E$, one can define a LD structure on V by

$$\mathcal{D} = \{(v, v^*); v \equiv \mathcal{L} v^* \pmod{E}\} \subset V \oplus V^*. \quad (4)$$

We first give the following lemma which is essential for the proof. It was originally given for Dirac structures in [1], and for LD structures the result was used in [7] without proof.

Lemma 1. *Let V be a n -dimensional vector space and V^* denote its dual space. A pairing on $V \oplus V^*$ given by*

$$\ll (v_1, v_1^*), (v_2, v_2^*) \gg = \langle v_1^* | v_2 \rangle + \langle v_2^* | v_1 \rangle - 2 \Psi(v_1^*, v_2^*), \quad (5)$$

for all $(v_1, v_1^), (v_2, v_2^*) \in V \oplus V^*$ where $\Psi : V^* \times V^* \rightarrow \mathbb{R}$ is a symmetric bilinear form, defines a non degenerate symmetric bilinear form of split sign.*

Proof. After choosing a proper basis for V , the result will be clear. First define the associated linear map $\psi : V^* \rightarrow V$ to Ψ by

$$\langle v_1^* | \psi v_2^* \rangle = \Psi(v_1^*, v_2^*) \text{ for all } (v, v^*) \in V \oplus V^*. \quad (6)$$

Then let $\alpha_1, \dots, \alpha_n$ be a basis of V and $\alpha_1^*, \dots, \alpha_n^*$ be a basis of V^* such that $\langle \alpha_i^* | \alpha_j \rangle = \delta_i^j$, $i, j = 1, \dots, n$, where δ is the Kronecker symbol. As a basis of $V \oplus V^*$ one can choose $(\alpha_1, 0), \dots, (\alpha_n, 0), (\psi \alpha_1^*, \alpha_1^*), \dots, (\psi \alpha_n^*, \alpha_n^*)$, then the matrix associated to the bilinear form in (5) becomes

$$\begin{pmatrix} \mathbf{0}_n & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{0}_n \end{pmatrix},$$

where $\mathbf{0}_n$ is the $n \times n$ zero matrix and \mathbf{I}_n is the $n \times n$ identity matrix. Accordingly, the basis given by

$$\begin{aligned} y_i &= \frac{\sqrt{2}}{2} ((\alpha_i, 0) + (\psi \alpha_i^*, \alpha_i^*)), \\ x_i &= \frac{\sqrt{2}}{2} ((\alpha_i, 0) - (\psi \alpha_i^*, \alpha_i^*)) \end{aligned}$$

gives the diagonal form

$$\begin{pmatrix} \mathbf{I}_n & 0 \\ 0 & -\mathbf{I}_n \end{pmatrix}.$$

Then it is concluded that the bilinear form in (5) has signature $\underbrace{\{+1, \dots, +1\}}_{n \text{ times}}, \underbrace{\{-1, \dots, -1\}}_{n \text{ times}}$. \square

Proof of Theorem 1. Existence of the map \mathcal{L} goes along with the lines with the usual argument [1] as follows. For every point $x = (v, v^*) \in V \oplus V^*$ the linear map \mathcal{L} is defined by $\mathcal{L} v^* = v|_{\rho^*(\mathcal{D})}$, where v is considered as a linear map on V^* . To show that it is well-defined, suppose we have $x = (v_1, v_1^*), x' = (v_2, v_2^*) \in \mathcal{D}$ with $v_1^* = v_2^*$. Then clearly $(v_1 - v_2, 0) \in \mathcal{D}$ which implies that $v_1 - v_2 \in \mathcal{D} \cap V$. By the condition (3), this is equivalent to saying that $(v_1 - v_2)|_{\rho^*(\mathcal{D})} = 0$ or $\mathcal{L} v_1^* = \mathcal{L} v_2^*$, as desired. The claim that the kernel of \mathcal{L} being $\mathcal{D} \cap V^*$ is trivial.

Conversely, suppose that a subspace $E \subset V$ and a linear map $\mathcal{L} : E^\circ \rightarrow V/E$ are given. Then, if \mathcal{D} is defined as in (4), it is easily seen that $\rho^*(\mathcal{D}) \subset (\mathcal{D} \cap V)^\circ$ is satisfied automatically. It remains to show \mathcal{D} has dimension n to satisfy the conditions of (ii) of Proposition 2. Since \mathcal{D} includes subspaces $E \oplus \{0\}$ and $\mathcal{L}(E^\circ) \oplus E^\circ$, it is of at least n -dimensional. Observe that \mathcal{L} can be extended to whole V^* , and this gives an symmetric bilinear form on V^* , defined by

$$\Psi(v_1^*, v_2^*) := \frac{1}{2} (\langle v_2^* | \mathcal{L} v_1^* \rangle + \langle v_1^* | \mathcal{L} v_2^* \rangle). \quad (7)$$

Then by Lemma 1, \mathcal{D} is a maximally isotropic subspace for an inner product on $V \oplus V^*$ of split sign. Eventually $\dim \mathcal{D} = n$. \square

Remark 1. If the linear map \mathcal{L} in associated to a LD structure \mathcal{D} is skew-symmetric, then the LD structure turns out to be a Dirac structure. But in general \mathcal{L} can be uniquely written as a sum of a skew-symmetric form \mathcal{L}^- and symmetric form \mathcal{L}^+ :

$$\mathcal{L} = \mathcal{L}^- + \mathcal{L}^+. \quad (8)$$

If \mathcal{L} is purely symmetric or skew-symmetric, then additionally one has

$$\rho(\mathcal{D})^\circ = \mathcal{D} \cap V^*. \quad (9)$$

To see this one can check that if \mathcal{L} is skew-symmetric then

$$\langle \rho^*(\mathcal{D}) | \rho(\mathcal{D} \cap V) \rangle = -\langle \rho^*(\mathcal{D} \cap V) | \rho(\mathcal{D}) \rangle = 0,$$

if \mathcal{L} is symmetric then

$$\langle \rho^*(\mathcal{D}) | \rho(\mathcal{D} \cap V) \rangle = \langle \rho^*(\mathcal{D} \cap V) | \rho(\mathcal{D}) \rangle = 0.$$

So, $\rho(\mathcal{D}) \subset (\mathcal{D} \cap V)^\circ$ in both cases. A dimension count gives the equality, since $\dim(\mathcal{D}) = n$.

The equations (3) and (9) are called the characteristic equations [1]. But the converse of the above claim is not generally true, that is, (3) and (9) are not sufficient for \mathcal{L} to be symmetric or skew-symmetric. For instance, if \mathcal{L} is defined on whole V^* and it is of full rank, then the characteristic equations are satisfied.

From the proof of Theorem 1, LD manifolds have the following property.

Corollary 1. *If \mathcal{D} is a LD structure on a vector space V , then there exists an inner product of split sign on $V \oplus V^*$ which respect to which \mathcal{D} is a Lagrangian subspace.*

Theorem 1 gives a representation of a LD structure, another representation can be given as follows [1].

Theorem 2. *Let \mathcal{D} be a LD structure on an n -dimensional vector space V , then there exist two linear maps $\mathbf{a} : \mathbb{R}^n \rightarrow V$ and $\mathbf{b} : \mathbb{R}^n \rightarrow V^*$ such that*

$$\ker \mathbf{a} \cap \ker \mathbf{b} = \{0\}, \quad (10)$$

$$(\mathbf{a}(\ker \mathbf{b}))^\circ \subset \text{range } \mathbf{b}. \quad (11)$$

Conversely, any LD structure \mathcal{D} on V can be defined by

$$\mathcal{D} = \{(\mathbf{a}x, \mathbf{b}x); x \in \mathbb{R}^n\} \subset V \oplus V^* \quad (12)$$

for two linear maps $\mathbf{a} : \mathbb{R}^n \rightarrow V$ and $\mathbf{b} : \mathbb{R}^n \rightarrow V^*$ satisfying (10) and (11).

Proof. Let \mathcal{D} be a LD structure on V . If one chooses a basis for D , then this is equivalent to giving two linear maps $\mathbf{a} : \mathbb{R}^n \rightarrow V$ and $\mathbf{b} : \mathbb{R}^n \rightarrow V^*$ such that the basis becomes $(\mathbf{a}e_1, \mathbf{b}e_1), \dots, (\mathbf{a}e_n, \mathbf{b}e_n)$. Since \mathcal{D} is n -dimensional, (10) is satisfied. Observe that $\mathcal{D} \cap V = \ker \mathbf{b}$ and $\rho^*(\mathcal{D}) = \text{range } \mathbf{b}$. Then by the defining property (3) of \mathcal{D} , the relation (11) is satisfied.

Conversely, assume that \mathcal{D} is given by (12), then by (10) it is n -dimensional, and (11) implies (3) by Proposition 2 (ii). \square

2.2 Leibniz-Dirac structures on manifolds

Let M be a n -dimensional smooth manifold. Consider a subbundle \mathcal{D} of the Pontryagin bundle $TM \oplus T^*M$. The following associated distributions and co-distributions to \mathcal{D} will be of importance throughout:

$$\begin{aligned} G_0 &= \{X \in TM; (X, 0) \in \mathcal{D}\}, \\ G_1 &= \{X \in TM; (X, \alpha) \in \mathcal{D} \text{ for some } \alpha \in T^*M\}, \end{aligned}$$

and

$$\begin{aligned} P_0 &= \{\alpha \in T^*M; (\alpha, 0) \in \mathcal{D}\}, \\ P_1 &= \{\alpha \in T^*M; (X, \alpha) \in \mathcal{D} \text{ for some } X \in TM\}. \end{aligned}$$

For simplicity we will assume G_0, G_1, P_0 and P_1 to be of constant rank. Recall that points on the distributions G_0, G_1 and the co-distributions P_0, P_1 have constant rank are called *regular points*. The set of regular points is an open dense set [1].

If there are more than one LD structures, we will denote the bundles associated to \mathcal{D} by $G_0(\mathcal{D})$, etc. For any distribution G the co-distribution $\text{ann}(G)$ is defined by

$$\text{ann}(G) = \{\alpha \in T^*M; \langle \alpha | X \rangle = 0 \text{ for all } X \in G\},$$

and for any co-distribution P the distribution $\ker(P)$ is defined by

$$\ker(P) = \{X \in TM; \langle \alpha | X \rangle = 0 \text{ for all } \alpha \in P\},$$

where $\langle | \rangle$ denotes the natural pairing between 1-forms and vector fields.

Definition 1 can be given on a manifold as the following.

Definition 2. Let \mathcal{D} be a smooth vector subbundle of $TM \oplus T^*M$. Then \mathcal{D} is called a *Leibniz-Dirac structure* (LD structure) if

$$P_1 = \text{ann } G_0. \quad (13)$$

The following result gives a representation of a smooth LD structure whose proof is analogous to the linear case when considered pointwise.

Theorem 3. If \mathcal{D} is a LD structure on an n -dimensional manifold, then there exists a vector bundle map $\mathcal{L} : P_1 \rightarrow TM/G_0$ for which

$$\mathcal{D} = \{(X, \alpha) : X \equiv \mathcal{L} \alpha \pmod{G_0}\}. \quad (14)$$

Conversely, if a distribution G_0 on M of constraint rank, and a vector bundle map $\mathcal{L} : \text{ann } G_0 \rightarrow TM/G_0$ are given, then a LD structure can be defined by (14).

Example 1. If the bundle map \mathcal{L} is skew-symmetric, then the LD structure turns out to be a Dirac structure. This case includes many geometries like symplectic, Poisson, foliation, etc.

Example 2. If there is a pseudo-Riemannian metric g on M , then it gives a bundle map $\mathcal{G} : TM \rightarrow T^*M$. Then the inverse, say, \mathcal{L} of this map gives rise to a LD structure which is symmetric. This setting allows one to define gradient control systems [7]. Note that one can also define a pseudo-Riemannian manifold as a Lagrangian subbundle with respect to the skew-symmetric pairing

$$\langle (X_1, \alpha_1), (X_2, \alpha_2) \rangle_- = \langle \alpha_1 | X_2 \rangle - \langle \alpha_2 | X_1 \rangle$$

for all $(X_1, \alpha_1), (X_2, \alpha_2) \in TM \oplus T^*M$ which was given originally in [1]. This will be studied in a future paper [15].

Example 3. In the general case, if \mathcal{L} is as in Theorem 3 and $P_1 = T^*M$, then LD manifold is simply called Leibniz manifold. These manifolds are studied in [8] but our definition also allows to add some constraints when modeling physical systems by LD structures (cf. Section 3).

As a linear LD structure, a smooth LD structure has the following property.

Corollary 2. If a manifold M has a LD structure, then \mathcal{D} is a Lagrangian subbundle with respect to a symmetric pairing on $TM \oplus T^*M$.

Remark 2. As in the linear case \mathcal{L} can be uniquely written as a sum of a skew-symmetric form \mathcal{L}^- and symmetric form \mathcal{L}^+ :

$$\mathcal{L} = \mathcal{L}^- + \mathcal{L}^+. \quad (15)$$

Observe also that \mathcal{L} can be extended locally to a bundle map from T^*M into TM which we also denote by \mathcal{L} . Thus one obtains associated tensor fields $\mathcal{B}^L : T^*M \times T^*M \rightarrow \mathbb{R}$ and $\mathcal{B}^R : T^*M \times T^*M \rightarrow \mathbb{R}$ by

$$\mathcal{B}^L(\alpha_1, \alpha_2) = \langle \alpha_2 | \mathcal{L} \alpha_1 \rangle \quad (16)$$

$$\mathcal{B}^R(\alpha_1, \alpha_2) = \langle \alpha_1 | \mathcal{L} \alpha_2 \rangle \quad (17)$$

for all $\alpha_1, \alpha_2 \in T^*M$. \mathcal{B}^L and \mathcal{B}^R are called left and right Leibniz tensors [8], respectively. Recall that there corresponds Leibniz parenthesis $[\cdot, \cdot]^L$ and $[\cdot, \cdot]^R$ associated to both Leibniz structures \mathcal{B}^L and \mathcal{B}^R , respectively, given by

$$[f, g]^L = \mathcal{B}^L(df, dg) \quad \text{and} \quad [f, g]^R = \mathcal{B}^R(df, dg). \quad (18)$$

In the next section we will define an associated bracket to these tensors.

Integrability of LD structures on manifolds is defined in accordance with the one on Dirac structures [1].

Definition 3. Let \mathcal{D} be a LD structure on a manifold M . If the characteristic distribution G_0 is integrable, then \mathcal{D} is called an integrable LD structure.

In fact, this is a conclusion of a Dirac structure being integrable but in this generalized case of LD structures, integrability definition makes sense. This can be seen by the following result.

Proposition 3. Let \mathcal{D} be an integrable LD structure on a manifold M . If the foliation of G_0 is denoted by Φ , then M/Φ inherits two Leibniz structures, namely, left and right Leibniz structures.

Proof. 1-forms on M/Φ are considered as Φ -invariant 1-forms $\alpha \in \Omega^1(M)$: $\alpha(T\Phi) = 0$. Then induced structures are defined by means of Φ -invariant 1-forms. They are apparently linear and well-defined, and so give Leibniz structures. \square

Remark 3. Recall that, in case of Dirac structures the induced Leibniz structures coincide and they are specifically Poisson structures [1].

3 Dynamics on LD manifolds

Dynamic properties of LD structures are given in this section. We first give the notion of admissible functions. The main ingredient of this section is a formulation of dissipative Hamiltonian systems with constraints.

3.1 Admissible functions

Admissible functions on LD manifolds are defined as in the Dirac case [1]. Let \mathcal{D} be a LD structure on a manifold M , and let $\Gamma(\mathcal{D})$ denote the smooth sections of \mathcal{D} . A function on M is called an *admissible function* if $df \in \Gamma(\mathcal{D})$. If f is an admissible function, then $(X_f, df) \in \mathcal{D}$ for some vector field X_f on M . Accordingly, two brackets on admissible functions on M can be defined by

$$[f, g]^L = X_f(g) \quad (19)$$

and

$$[f, g]^R = X_g(f). \quad (20)$$

As usual let \mathcal{L} denote a representation of \mathcal{D} and let \mathcal{B}^L and \mathcal{B}^R denote the left and right LD tensors respectively (cf. Remark 2). Then

$$[f, g]^L = \mathcal{B}^L(X_f, X_g) \quad (21)$$

and

$$[f, g]^R = \mathcal{B}^R(X_f, X_g). \quad (22)$$

The following result is an extension of the Dirac case [1]. We only consider left Leibniz bracket, the same result follows for the right Leibniz bracket in a similar way.

Proposition 4. *With the notation above, admissible functions satisfy the Leibniz identity with respect to the left Leibniz bracket:*

$$[fg, h]^L = g[f, h]^L + f[g, h]^L. \quad (23)$$

Furthermore, if f and g are admissible functions, so is the product fg .

Proof. Let (X_f, df) and (X_g, dg) be two sections of \mathcal{D} . Then

$$\begin{aligned} [fg, h]^L &= X_{fg}(h) = gX_f(h) + fX_g(h) \\ &= g[f, h]^L + f[g, h]^L \end{aligned}$$

which proves the Leibniz identity. To prove the other claim one computes

$$\begin{aligned} (X_{fg}, d(fg)) &= (gX_f + fX_g, gdf + fdg) \\ &= g(X_f, df) + f(X_g, dg) \in \Gamma(\mathcal{D}). \end{aligned}$$

So, fg is an admissible function. □

From the above result, admissible functions on M form a Leibniz bracket which was defined in [9] not to be confused with Leibniz algebra brackets [16].

Remark 4. *Recall the fact that the bracket of two admissible functions is again an admissible function, which follows from the Jacobi identity. Nevertheless, this may not hold in a manifold with a general LD structure.*

Remark 5. *Observe that, a proof of Proposition 3 can be given in terms of admissible functions as originally given in the Dirac case [1].*

3.2 Dissipative Hamiltonian systems with constraints

Let \mathcal{D} be a LD structure on a manifold M , then one can extend the notion of implicit Hamiltonian systems [17] to (M, \mathcal{D}) as follows.

Definition 4. *Let $H : M \rightarrow \mathbb{R}$ be a Hamiltonian. The dissipative implicit Hamiltonian system (DIHS) corresponding to (M, \mathcal{D}, H) is given by*

$$(\dot{x}, dH(x)) \in \mathcal{D}(x), \quad x \in M. \quad (24)$$

In this setting, G_1 describes the set of admissible flows and P_1 describes the set of algebraic constraints.

If the splitting (15) is considered, by (26) one obtains

$$\frac{dH}{dt} = \langle \frac{\partial H}{\partial x}(x) | \mathcal{L}^-(x) \frac{\partial H}{\partial x}(x) \rangle. \quad (25)$$

This equation has several meanings by the following specific examples.

Example 4. *If the bundle map \mathcal{L} is skew-symmetric, then the GIHS in (24) is an implicit Hamiltonian system and the equation (25) gives the conservation of energy. This special case of conservative Hamiltonian systems with constraints have been studied by many authors in Dirac structures framework [18, 17, 5, 6].*

Example 5. *If \mathcal{L} is symmetric then the system (24) is a generalization of pseudo-gradient systems [7]. This can be seen as follows. If \mathcal{L} is taken as the restriction to TM/G_0 of inverse of the musical isomorphism $g^\flat : TM \rightarrow T^*M$ of a pseudo-Riemannian metric g on M , then one obtains a gradient system with constraints. In that case the equation (25) is called the entropy equation [7]. As was pointed out in Exapmle 2, it is more advantageous to handle these systems in terms of Beltrami-Dirac structures [15].*

Example 6. *If \mathcal{L} is arbitrary, that is, not necessarily symmetric or skew-symmetric, then the GIHS is a generalization of metriplectic systems with constraints [19]. In this case the equation (25) is may be considered as the dissipation equation [19]. This general case is used to model so many physical systems. For example, systems with nonholonomic constraints, dissipation, etc [20, 21, 10, 8, 22, 13, 23, 11].*

Remark 6. *Assume that \mathcal{D} is represented in the form (14), then the DIHS corresponding to (M, \mathcal{D}, H) has a local representation*

$$\begin{aligned} \dot{x} &= \mathcal{L}(x) \frac{\partial H}{\partial x}(x) + g(x) \lambda, \\ 0 &= g^T(x) \frac{\partial H}{\partial x}(x), \end{aligned} \quad (26)$$

where $\frac{\partial H}{\partial x}(x)$ stands for the column vector of partial derivatives of H , and $g(x)$ is a full rank matrix with $\text{Im } g(x) = G_0(x)$, and λ are Lagrange multipliers corresponding to the algebraic constraints $0 = g^T(x) \frac{\partial H}{\partial x}(x)$ [17].

Example 7. Let Q be an n -dimensional configuration space and let $q = (q_1, \dots, q_n)$ be a local coordinate system on Q . Kinematic constraints on Q are given by

$$\mathbf{A}^T(q) \dot{q} = 0, \quad (27)$$

where $\mathbf{A}(q)$ is an $n \times k$ matrix, $k \leq n$, whose rank is k for all $q \in Q$. Then the constrained dissipative Hamiltonian system on T^*Q is defined by equations

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \underbrace{\begin{bmatrix} \mathbf{O}_n & \mathbf{I}_n \\ -\mathbf{I}_n & \mathbf{R}(q) \end{bmatrix}}_{\mathcal{L}} \begin{bmatrix} \frac{\partial H}{\partial q}(q, p) \\ \frac{\partial H}{\partial p}(q, p) \end{bmatrix} + \begin{bmatrix} \mathbf{O}_n \\ \mathbf{A}(q) \end{bmatrix} \lambda, \\ 0 &= \begin{bmatrix} \mathbf{0}_n & \mathbf{A}^T(q) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q, p) \\ \frac{\partial H}{\partial p}(q, p) \end{bmatrix}, \end{aligned}$$

where $\mathbf{R}(q)$ is a semidefinite matrix.

Example 8. ([24]) We consider a particle moving on a surface $\dot{z} = y\dot{x}$ in \mathbb{R}^3 with a friction force proportional to the particle velocity. The Hamiltonian is given in terms of cartesian coordinates x, y, z and their conjugate momenta by

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2), \quad (28)$$

and the characteristic distributions are given by

$$\begin{aligned} G_0 &= \text{span}\left\{\frac{\partial}{\partial p_z} - y\frac{\partial}{\partial p_x}\right\} \\ P_1 &= \text{span}\{dx, dy, dz, ydp_z + dp_x, dp_y\}. \end{aligned}$$

The corresponding LD structure \mathcal{L} in matrix form is given by

$$\mathcal{L} = \mathbf{J} - \mathbf{R},$$

where

$$\begin{aligned} \mathbf{J} &= \begin{bmatrix} \mathbf{O}_3 & \mathbf{I}_3 \\ -\mathbf{I}_3 & \mathbf{O}_3 \end{bmatrix} \\ \mathbf{R} &= \begin{bmatrix} \mathbf{O}_3 & \mathbf{O}_3 \\ \mathbf{O}_3 & \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \end{bmatrix} \end{aligned}$$

with $\mu_i(q) > 0$ is the directional and space-dependent damping coefficient [10]. Then the equations of motion read

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & \mu_1 & 0 & 0 \\ 0 & -1 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & -1 & 1 & 0 & \mu_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ p_x \\ p_y \\ p_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ y \\ 0 \\ 1 \end{bmatrix} \lambda, \\ 0 &= yp_x - p_z. \end{aligned}$$

4 Reduction and transformation of LD structures

In this section, we first discuss reduction of LD structures on subspaces and more generally on subbundles. Then in Subsection 4.2, we consider maps which push forward LD structures.

4.1 Reduced LD structures

Let \mathcal{D} be a LD structure on a vector space V . If $\Sigma \subset V^*$ is a subspace, then it is possible to reduce the LD structure on Σ . For this purpose, it is desirable to use the representation of the LD structure given in Theorem 1.

If \mathcal{D} is given by a subspace $E \subset V$ a linear map $\mathcal{L} : E^\circ \rightarrow V/E$, then the reduced LD structure \mathcal{D}_Σ is defined by the subspace $E_\Sigma = \ker(E^\circ \cap \Sigma)$ and the restriction \mathcal{L}_Σ of \mathcal{L} to $E^\circ \cap \Sigma$. Then \mathcal{D}_Σ is given by

$$\mathcal{D}_\Sigma = \{(w, w^*); w \equiv \mathcal{L}_\Sigma w^* \pmod{E_\Sigma}\} \subset \Sigma^* \oplus \Sigma. \quad (29)$$

Observe that an alternative way of defining reduced Dirac structures on subspaces of V is not trivial in general. Because, this makes sense if linear maps of form $\omega : V \rightarrow V^*$ exist. However, if \mathcal{L} is symmetric or skew-symmetric one can also define a linear map from a proper subset of V into V^* and the procedure of reduction given in [1] works.

The construction of reduction given in the linear case can also be considered for smooth LD structures. If \mathcal{D} is a smooth LD structure on a manifold M , and if $\Sigma \subset T^*M$ is a smooth submanifold, then the following *clean intersection* property is required to reduce \mathcal{D} . Let us denote the reduced LD structure by \mathcal{D}_Σ . As the characteristic distribution $P_1(\mathcal{D}_\Sigma)$ of \mathcal{D}_Σ will be $P_1(\mathcal{D}) \cap T^*M$, this distribution should be a subbundle. Then one can define the reduced LD structure as an extension of (29).

4.2 LD maps

In this subsection, we study maps which carry LD structures to LD structures. We start with the linear case then consider smooth LD maps. Let \mathcal{D} be a LD structure on a vector space V . As the LD structure \mathcal{D} is associated to a linear map $\mathcal{L} : V^* \rightarrow V$, we can be pushed forward \mathcal{D} into another vector space by a linear map [14].

Definition 5. Let $\phi : V \rightarrow W$ be a linear map between vector spaces, and let \mathcal{D}_V be a LD structure on V . Then push forward image of \mathcal{D}_V is defined by

$$\mathcal{F}_\phi(\mathcal{D}_V) = \{(\phi v, w^*); v \in V, w^* \in W^*, (v, \phi^* w^*) \in \mathcal{D}_V\} \subset W \oplus W^*, \quad (30)$$

where $\phi^* : W^* \rightarrow V^*$ denotes the adjoint map of ϕ .

Proposition 5. Let $\phi : V \rightarrow W$ be a linear map. If \mathcal{D} is a LD structure on V , then $\mathcal{F}_\phi(\mathcal{D}_V)$ is a LD structure on W .

Proof. Let us denote the subspace $\mathcal{F}_\phi(\mathcal{D}_V)$ by \mathcal{D}_W , and let ρ_V, ρ_V^* (resp. ρ_W, ρ_W^*) denote the projections from $V \oplus V^*$ (resp. $W \oplus W^*$) onto the first and the second factors respectively. First notice that

$$\mathcal{D}_W \cap W = \phi(\mathcal{D}_V \cap V). \quad (31)$$

Let us consider the associated linear map $\mathcal{L}_V : P_1(\mathcal{D}_V) \subset V^* \rightarrow V$ to \mathcal{D}_V which is defined in Theorem 1. Then there is a well-defined map \mathcal{L}_W from $\rho_W^*(\mathcal{F}_\phi(\mathcal{D}_V))$ into W given by

$$\mathcal{L}_W w^* := \phi(\mathcal{L}_V(\phi^* w^*)). \quad (32)$$

We claim that $\mathcal{F}_\phi(\mathcal{D}_V) = \mathcal{D}_W$ where

$$\mathcal{D}_W = \{(w, w^*); w \equiv \mathcal{L}_W w^* \pmod{D_W \cap W}\}. \quad (33)$$

which is then a LD structure on W by Theorem 1. We first show that $\mathcal{F}_\phi(\mathcal{D}_V) \subset \mathcal{D}_W$. In fact, $(w, w^*) \in \mathcal{F}_\phi(\mathcal{D}_V)$ implies $w = \phi v$ with $(v, \phi^* w^*) \in \mathcal{D}_V$, then

$$v \equiv \mathcal{L}_V(\phi^* w^*) \pmod{(\mathcal{D} \cap V)} \quad (34)$$

by the definition. This shows, by considering the images of both sides of the equation (34) under ϕ , that

$$w \equiv \mathcal{L}_W w^* \pmod{D_W \cap W}$$

by (31) and (32). Then $(w, w^*) \in \mathcal{D}_W$. To show that $\mathcal{D}_W \subset \mathcal{F}_\phi(\mathcal{D}_V)$ we assume that $(w, w^*) \in \mathcal{D}_W$, namely $w \equiv \mathcal{L}_W w^* \pmod{D_W \cap W}$. This implies by (31) and (32) that $w = \phi(\mathcal{L}_V(\phi^* w^*)) + \phi w'$ for some $w' \in \mathcal{D}_V \cap V$. Since ϕ is linear $w = \phi v$ for $v = \mathcal{L}_V(\phi^* w^*) + w'$ which implies that $(v, \phi^* w^*) \in \mathcal{D}_V$. Then $\mathcal{F}_\phi(\mathcal{D}_V) \subset \mathcal{D}_W$. This concludes the proof. \square

By the proof of the preceding result the following conclusion is obtained as in the Dirac case [14].

Corollary 3. *Let $\phi : V \rightarrow W$ be a linear map, and \mathcal{D} be a LD structure on V , Then for the Dirac structure $\mathcal{D}_W = \mathcal{F}_\phi(\mathcal{D}_V)$ on W the following equalities hold:*

$$\mathcal{D}_W \cap W = \phi(\mathcal{D}_V \cap V) \quad (35)$$

and

$$\mathcal{L}_W = \phi \circ \mathcal{L}_V \circ \phi^*. \quad (36)$$

Remark 7. *Alternatively, one can follow the methods given in [14] to study maps between vector spaces with LD structures. As indicated in [14], the setting for the Lagrangian relations [25] is valid between inner products of split sign on $V \oplus V^*$ and $W \oplus W^*$. By using Corollary 1, one can set suitable inner products on both spaces and an alternative proof of Proposition 5 can be given.*

Now we proceed with the smooth case. Let M be a manifold and let $\mathcal{D}_M \subset TM \oplus T^*M$ be a LD structure. If $\phi : M \rightarrow N$ is a smooth map then the *forward image* of \mathcal{D}_M is defined by

$$\mathcal{F}_\phi(\mathcal{D}_M) = \{(\phi_*(X), \beta); (X, \phi^*(\beta)) \in \mathcal{D}_M\} \quad (37)$$

Clearly, $\mathcal{F}_\phi(\mathcal{D}_M)$ may not be a subbundle, to ensure its being a subbundle the following definition is given. \mathcal{D}_M is called *ϕ -invariant* if

$$\mathcal{F}_\phi(\mathcal{D}_M)_x = \{(\phi_*(X), \beta) : (X, \phi^*(\beta)) \in (\mathcal{D}_M)_x\} \quad (38)$$

is invariant for all x on the same ϕ -fibre [26]. Nevertheless, ϕ -invariance is not sufficient, one also needs some extra assumptions as in the following [26].

Proposition 6. *Let $\phi : M \rightarrow N$ be a surjective submersion and \mathcal{D}_M be a LD structure on M . If $\ker(\phi_*) \cap \mathcal{D}_M$ has constant rank and \mathcal{D}_M is ϕ -invariant, then the forward image of \mathcal{D}_M defines a LD structure on N .*

Proof. First, denote $\mathcal{F}_\phi(\mathcal{D}_M)$ by \mathcal{D}_N , then $G_0(\mathcal{D}_N) = \phi_*(G_0(\mathcal{D}_M))$. As a result the rank of $G_0(\mathcal{D}_N)$ is constant if and only if the rank of $\ker(\phi_*) \cap \mathcal{D}_M$ is constant, since $G_0(\mathcal{D}_M)$ is a subbundle by the hypothesis. Thus \mathcal{D}_N is a subbundle. Next, by the ϕ -invariance, one can define a tensor field $\mathcal{L} : T^*N \rightarrow TN$ by

$$\mathcal{L}_N(\phi(x))\eta := \phi(\mathcal{L}_M(x)(\phi^*\eta)). \quad (39)$$

Then $\mathcal{F}_\phi(\mathcal{D}_M)$ can be defined in terms of \mathcal{L}_N as in Proposition 5. \square

Remark 8. *As it is pointed out in [26], if there is a Lie group action on M , which preserves the LD structure, then the conditions in the hypothesis of Proposition 6 are satisfied if furthermore the action is free and proper. This and related questions about reduction will be the subject of a future work.*

5 Conclusions

We defined linear and smooth Leibniz-Dirac structures which are generalizations of Dirac structures. We studied the geometry and dynamics of LD structures both on linear spaces and manifolds. It was explained with several examples that LD structures are capable of formulating dissipative implicit Hamiltonian systems with constraints. We hope that LD structures will find applications in physics and related area.

However, there remain many questions on both geometric and dynamics properties of LD structures unexplored, which we address herewith. As it is known, a more general setting of Dirac structures on Courant algebroids has a growing importance [27, 28]. Accordingly, LD structures may be defined on generalizations of Courant algebroids such as Leibniz algebroids [29, 30]. As also pointed out in Remark 8, one can study reduction of symmetric LD structures as a generalization of Dirac reduction [28, 31]. Another related problem is quantization of dissipative systems [32]. We hope our results will contribute to this and related problems on dissipative Hamiltonian systems with constraints.

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